Planar motion analysis (060211)

Problem An homography (projective transform) which transforms the points of a plane to points on the same plane is a linear transform in 2D homogeneous coordinates. Let the matrix H define that transform.

A point is said to be fixed for H if it transforms to itself.

a) How many fixed points are in a generic planar homography H? Why? (3 pts)

b) Isometry (rigid displacement) is a special homography which keeps distances and angles. A rigid planar displacement can always be reduced to a single rotation about a specific center of rotation; then, how many fixed points are in a generic isometry?

c) Suppose that the plane π on which the rigid displacement (described by an isometry H) takes place, is observed by a non calibrated, fixed camera. Let T be the matrix descibing the projective transform between the motion plane and the image plane. Let L be the transform relating the image of a generic point of π before the transform to its image after the transform. How do you write L as function of T and H? (2 pts)

d) What about the fixed points of L? (3 pts)

e) Consider a planar quadrangle, whose shape is unknown, lying on a plane π. It is subject to an unknown planar rigid displacement. The scene is imaged by a fixed, non calibrated camera in an unknown position. Describe a technique for reconstructing the shape of the quadrangle and its displacement, up to a scale factor. (9 pts)

Solution

a) If the transformed point, whose expression is $x' = Hx$ coincides with x, then, since we are in homogeneous coordinates, $kx = Hx$. That is an equation at eigenvectors and eigenvalues. Then the fixed points – the invariants for the homography H – are H's eigenvectors. Since H is 3x3 square, there are in general 3 eigenvectors, that is, 3 invariants, that is 3 fixed points.

b) A planar isometry is a rotation around a specific center of rotation. Then the center of rotation is invariant. The other two invariants are the circular points (they belong to the infinite line).

c) $L = THT^{-1}$

d) Since the center or rotation and the circular points do not change because of the rigid displacement H, then also their images do not change. Since the two circular points define the infinite line, the infinite line and its image are invariant as well (but note that other points on the infinite line are not invariant as they are transformed to different points on the infinite line). This can also be found analytically.

e) Consider $(A', B'.C'.D')$, image of the planar quadrangle (A, B, C, D) before the displacement; and $(A'', B''.C''.D'')$, image of the quadrangle after the displacement. The image of the infinite line is known.

The intersections between the image of the infinite line and the line (on the image) on which an edge (e.g. $A'B'$) lies is a vanishing point of the edge direction. There are 8 edges, so 8 vanishing points can be found. Their directions depend on 4 unknowns: the three internal angles of the quadrangle (A, B, C, D) and the rotation angle θ of the rigid displacement (as a reference, assume that 0 is AB's direction before the diplacement).

Since 8 vanishing points allow to write 5 independent equations enforncing cross-ratio invariance, the 4 unknown angles can be found. Therefore, we can know, for every vanishing point on the infinite line, the associate direction can be determined.

Let AB's length be 1 (this fixes the scale of the reconstruction); the shape of (A, B, C, D) is found as follows: the vanishing points of the directions of the diagonals AC and BD are found by intersecting the image lines $A'C'$ and $B'D'$ with the image of the infinite line: since we know the direction of each vanishing point on the infinite line, we know the directions of AC and BD. Knowing the internal angles and the diagonal directions, the shape of the quadrangle is determined.

The only remaining unknowns are the traslation parameters of the rigid displacement (we have determined θ, but not the center of rotation). They are found, for example, by determining the displacement vector of A. You can consider a reference frame centered on A before the displacement, and oriented in such a way that B's coordinates (before the displacement) are $[1, 0]$. In this new reference frame, in order to find the coordinates of A after the displacement you only need to consider two directions, which can be determined from their vanishing points on the infinite line:

- the direction defined by A before the displacement and A after the displacement, and
- the direction defined by B before the displacement and A after the displacement.

Vertical poles

Problem Two vertical poles (not necessarily of the same length) are placed over an horizontal plane. The poles project a shadow on the plane (the scene is lit by the sun). The scene is imaged by an uncalibrated perspective camera.

You want to alter the image, inserting a third, virtual vertical pole. You must make sure that its direction is correct. You also want to draw its shadow, with correct direction and length.

Solution The direction of the third pole in the image is easily determined: the two existing poles are parallel, and in the image the intersection of their lines allows to determine the vertical vanishing point v_v. The line on which the third pole lies must pass through v_v.

The endpoints of the virtual pole are not constrained since its position the plane and length can be chosen freely.

The direction of the shadow is determined in a similar way. The two existing shadows are parallel in the scene (because the sun is at infinite distance). By intersecting their lines in the image we find an horizontal vanishing point v_h. The shadow of the virtual pole must lie on a line passing through v_h.

One endpoint of the new shadow is fixed at the base of the new pole. The other endpoint is *not* free, but depends on the direction of the sun. In particular, the vanishing point of the sun's direction (name it v_s) is found in the image by intersecting the lines connecting the head of each pole with its shadow. The other endpoint of the virtual shadow lies on the image line defined by the head of the virtual pole and v_s.

Note: v_h, v_s and v_v are collinear. Why?

Calibration of a natural camera from two coaxial circumferences (050218)

See word file

Calibration from a cube (060201)

See word file

A circle and its center (050906)

See web page: `http://www.leet.it/home/lale/joomla/content/blogcategory/14/41/`
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Contour of a cone (060703, part C)

Question 1

Problem What is the absolute conic? Why is it invariant under similarity transforms? Write a constraint about the image of the absolute conic, which can be derived from the knownledge of two vanishing points associated to orthogonal directions.

Solution The absolute conic is a conic of complex points given as the intersection of any sphere and the infinite plane.

A similarity transform transforms a sphere to another sphere; so its intersection with the infinite plane – the absolute conic – does not change. Even the image of the absolute conic is therefore constant when similarities are applied to the world w.r.t. the camera; then the image of the absolute conic is invariant when the camera is moved or rotated. That is, it only depends on the intrinsic camera parameters.

Consider a sphere and a point P on the infinite plane. The polar plane of P w.r.t. the sphere is perpendicular to P's direction and contains the center of the spehere. If we intersect this with the infinite plane, in the infinite plane the plar line of P w.r.t. the absolute conic (intersection between the sphere and the infinite plane), coinncides with the intersection between the polar plane and the infinite plane. This intersection is the line of infinite points orthogonal to the direction of the infinite point P. Therefore infinite points conjugate w.r.t. the absolute conic are pairwise orthogonal, and vice-versa. This translates on the image (omography from the infinite plane to the image plane) as follows: vanishing points in orthogonal directions are conjugate w.r.t. the image of the absolute conic; this originates the usual constraint.

Question 2

Problem Consider a degenerate quadric – more precisely, a cone whose axis is not orthogonal to its base; its base is an ellipse and its vertex is at a generic position w.r.t. the base. What's the shape of the image contour of that cone?

Solution A meridian is a line contained in the cone and passing through the vertex. The normal to the cone at a point belonging to a meridian is obviously orthogonal to the meridian; moreover, for each of the meridian's points the normal to the cone have the same direction. Moreover, given a point on the meridian, the plane tangent to the cone passing through that point is perpendicular to the normal to the cone through that point. Moreover, since the meridian through that point is perpendicuar to the normal to the cone, the meridian is contained in the tangent plane to the cone through that point. Then, all the points on the meridian through P are contained in the tangent plane to the cone through P. If the viewpoint O is on a tangent plane, then all the points on the meridian belong to the tangent plane. Then the contour generator curve contains that meridian, and the apparent contour (its image) will contain the image line of the meridian. Therefore the apparent contour is a set of lines (possibly with the addition of part of the base if it is visible). These lines are two because, since the cone is a quadric, the apparent contour of a quadric is a conic, and a (non degenerate) conic is made by at most two lines.

Two mirrors

Given a point P and a planar mirror, a virtual point Q exists which is the reflextion of P. Q is placed in such a way that P and Q are symmetrical w.r.t. the mirror plane. In particular:

- the middle point of segment PQ lies on the mirror surface, and
- the direction of segment PQ is orthogonal to the mirror plane.

Two planar mirrors are given, with unknown position and orietation. Given a set of $n \geq 2$ points P_1, P_2 ... P_n in 3D space, consider their reflections Q_1, Q_2, ... Q_n generated by the first mirror and R_1, R_2, ... R_n generated by the second mirror. Ignore multiple reflections.

Using a calibrated camera, consider the images P'_1, P'_2 ... P'_n of the n real points and the images Q'_1, Q'_2, ... Q'_n, R'_1, R'_2, ... R'_n of the $2n$ virtual points.

You want to reconstruct the 3D shape of the set of points P_1, P_2 ... P_n.

Delineate the solution by following the given sequence of steps. Assume that the intersection between the two mirrors defines the vertical direction (z axis), whereas the x axis is the only horizontal line belonging to the first mirror.

a) Given a calibrated camera and the image of a parallelogram, say how the parallelogram image allows you to find the angle between the two couples of parallel edges and the length ratio of the parallelogram edges.

b) Show how you can retrieve the vanishing points (and the directions, since the camera is calibrated) of the directions normal to the two mirrors, and the image of the infinite line on the horizontal plane (which is the plane containing both normal directions to the mirrors).

c) Explain how you can recover the images of the orthogonal projections of the P_1, P_2 ... P_n points on each of the two mirrors. Note that the orthogonal projection of P on the first mirror is the middle point of the PQ segment, which connects point P with its reflection Q.

d) Using the camera calibration matrix, explain how you can determine the vanishing point of the vertical direction (z axis), and the vanishing points of the two horizontal directions, each contained in one of the two mirrors. One of the two horizontal directions, contained in the first mirror, is the x axis direction,.

e) Explain how, for each couple of points P_1, P_2, you can find the image of the rectangle specified as follows: the rectangle is completely containted in the first (or second) mirror, has a vertical edge belonging to the intersection line between the two mirrors, has an horizontal edge with an endpooint on P_1's projection on the mirror, and the other horizontal edge passes through the orthogonal projection of P_2 on the mirror.

f) Explain how, using the result of point a, you can reconstruct the set of points P_1, P_2 ... P_n, up to an unknwon scale factor, by building the images of an appropriate set of parallelograms

Solution

a) You can find the vanishing point of the directions of the edges of a parallelogram: just intersect the image lines on which two parallel edges lie. You can build the plane parallel to the interpretation lines of the vanishing points. By intersecting this plane with the viewing rays you obtain a parallelogram, similar to the original one, from which length ratios and angles can be retrieved.

b) The lines connecting any point P to its reflection Q are perpendicular to the first mirror. By intersecting the images of two of these connecting lines you obtain the vanishing point of the normal direction to the first mirror. The same can be done for the second mirror. By connecting the two obtained vanishing points, you get the infinite line of an horizontal plane (because it is the plane on which both normals to the mirrors lie).

c) The orthogonal projection K of a point P on the first mirror is the middle point of the segment connecting P to its reflection Q. These three collinear points and the infinite point along the same line are an harmonic quartet (i.e., their cross ratio is -1 if taken in the right order). Since the cross ratio is invariant between the scene and the image, the images of these four points (which are collinear) must be an harmonic quartet as well: this allows you to decide where the image of K lies, because you already see the image of P, the image of Q and the image of the infinite point.

d) The vanishing point of the vertical direction is just the projection of the 3D point (in homogeneous coordinates): $[0, 0, 1, 0]$, wheras the vanishing points of the horizontal directions contained in the two mirrors are the projections of $[1, 0, 0, 0]$ (because the first mirror contains the x axis), and $[1, \tan\alpha, 0, 0]$, where α is the angle between the two mirrors (found at point a).

e) The horizontal parallelogram will have an edge having P as first vertex and a point A belonging to the first mirror as second vertex. A will lie on the horizontal line parallel to the second mirror and passing through P; also, A lies on the horizontal line parallel to the first mirror passing through the orthogonal projection of P on the first mirror. Since the vanishing points of these directions are known, and the image of the orthogonal projection is known as well, the image A' of A can be found as the intersection of image lines. Similarly, you can find the image B' of the second vertex B of the second edge passing through P. The opposing vertex C, which lies at the intersection between the two mirrors, can be found by intersecting the edges parallel to the two edges found so far, passing through vertexes A and B. Then the image C' of C is found by intersecting the images of these edges, which can be found as the lines connecting A (B) with the appropriate vanishing point.

f) Once the parallelograms of the previous point have been built for two points P_1 and P_2, you have the vertexes ${C_1}'$ and ${C_2}'$, which lie on the image of the intersection between the two mirrors. So the considered rectangle passes through the orthogonal projection of P_1 on the first mirror and through points C_1 and C_2. The image of the missing vertex of the rectangle can be found easily, since the images ${C_1}'$ and ${C_2}'$ of C_1 and C_2 are already known, and the vanishing points of the directions of the two missing edges are known as well.

g) Since the orientations of the planes containing the parallelograms introduced at point e are known; and the orientations of the rectangles introduced at point f are known; and the images of these parallelograms and rectangles are known as well; then, using the results of point a, you can determine all the length ratios and angles. By fixing any length as a scale factor, you can reconstruct the 3D space coordinates of P_1, P_2 ... P_n relative to each other.

Reconstruction of the trajectories of falling bodies (060717)

Problem The trajectory of a body subject to gravity is a parabola with vertical axis (ignoring air viscosity). By observing the trajectories of a number ($n \geq 2$) of bodies, we want to determine the relative orientations between the planes which contain the trajectories; moreover, for each body, we want to compute the equation of the parabola.

a) Write down the matrix C represeting a parabola with vertical axis and determine:

- The intersection points between the parabola and the infinite line along the plane on which the parabola lies;
- The polar line of one of these intersection points w.r.t. the parabola.

b Given the projection matrix $P = [M|m]$ of a calibrated camera, and given the conics C'_i (with $i = 1..n$) representing the images of the described parabolic trajectories; describe a procedure for determining, for each body i,

- the orientation of the plane containing i's trajectory w.r.t. the camera,
- the equation of the parabola describing i's trajectory on the plane which contains it (up to a scale factor).

Suggestion: build upon the results of point a.

Solution In point a we find that the intersection between a parabola and the infinite line is a double point at infinite in the direction of the parabola's axis. Therefore, all parabolae with vertical axis intersect the infinite plane in the infinite point in vertical direction. Moreover, we show that the polar line of that infinite point w.r.t. the parabola is the infinite line of the plane containing the parabola.

Therefore you find the vertical direction, in the camera reference, as the interpretation line of the intersection point v between the images of the parabolae ($d_{\text{vert}} = M^{-1}v$). The polar line of that vanishing point v w.r.t. the conic C' (image of the parabolic trajectory of one of the bodies), given by $l = C'v$, is therefore the image of the infinite line along the plane containing the parabola described by that body. The direction of the plane containing that parabola, with respect to the camera, is easily found because it is parallel to the interpretation plane of the line l.

The equation of the parabola in the plane which contains it can be determined, after fixing a reference frame on that plane, in different ways: for example:

1. by intersecting the interpretation cone of the conic C' with an arbitrary plane (which determines the scale factor) parallel to the interpretation plane of the line l, or
2. by reprojecting C' on an image plane parallel to the interpretation plane of line l.

Triangle and triangular prism (060214)

Problem Consider a generic triangle (A, B, C) lying on a plane π. Near, there is a prism with triangular base; the base (A', B', C') of the prism is an exact copy of the (A, B, C) truabgke. Also the base (A', B', C') lies on π. The position and relatigve orientation of the two triangles are unconstrained and unknown.

Find a procedure for finding an euclidean reconstruction of pi, that is, for determining the shape of the two triangles (i.e. their internal angles), and their relative position and orientation.

Solution The directions of edges AB, BC and CA are coplanar, therefore the vanishing points of these directions are aligned. These vanishing points can be found by intersecting the image of each edge of the prism's bottom base, with the associated edge of the top base of the prism. The line containing the three vanishing points is the image of the infinite line, associated to the plane containing the prism's base.

Other three vanishing points, aligned to the previous ones (because they are associated to directions coplanar to those previously used) are associated to edges $A'B'$, $B'C'$ and $C'A'$. These vanishing points are found by intersecting each edge with the image of the infinite line found previously.

You now have six aligned vanishing points, which allows to write 3 independent cross-ratios. The angular unknowns to be determined are three: two of the three internal angles of the ABC triangle, and the rotation angle of the triangle $A'B'C'$ w.r.t. ABC. The unknowns are therefore foud by solving a system of three equations. The direction associated to any vanishing point on the image of the infinite line is therefore determined.

Once the angles are found, a reference on the plane can be determined by fixing its origin in A and its x axis along edge AB, and fixing AB as unitary length. For each point P of the plane the coordinates in this reference frame can be found by considering the two lines AP and BP; you can determine the direction of these lines by considering their intersection with the infinite line. Now, since, the direction of these lines are known, P's coordinates are known – because the lines connect P to known points.

This is an euclidean reconstruction of π.