

A SIMPLE PROOF OF THE CONLEY CONJECTURE FOR HAMILTONIAN DIFFEOMORPHISMS C^1 -CLOSE TO THE IDENTITY

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ABSTRACT. We provide a new simple proof of the Conley conjecture for standard symplectic tori, asserting that each Hamiltonian diffeomorphism ϕ admits infinitely many periodic points corresponding to contractible periodic orbits, under the further assumption that ϕ is C^1 -close to the identity. Our argument is based on generating functions.

RÉSUMÉ. On donne une nouvelle démonstration simple de la conjecture de Conley pour les tores symplectiques standards, affirmant que chaque difféomorphisme hamiltonien ϕ a une infinité de points périodiques correspondants à des orbites périodiques contractibles, sous l'ultérieure hypothèse que ϕ est C^1 -proche de l'identité. Notre argument est basé sur les fonctions génératrices.

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1. INTRODUCTION

A remarkable feature of Hamiltonian systems on symplectic manifolds is that they tend to have “many” periodic orbits. This is somehow a consequence of the fact that periodic orbits can be detected by a variational principle: they are critical points of the action functional. In 1984, Conley [Con84] conjectured that each Hamiltonian diffeomorphism of the standard symplectic torus $(\mathbb{T}^{2d}, \omega)$ has infinitely many contractible periodic points (a periodic point is contractible when it corresponds to a contractible periodic orbit of any associated Hamiltonian system,

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see Section 2.1). The weaker version of this conjecture, for a *generic* Hamiltonian diffeomorphism, was soon established by Conley and Zehnder [CZ84] and then extended to all the closed symplectically aspherical manifolds by Salamon and Zehnder [SZ92]. The general Conley conjecture is sensibly harder. In dimension 2, for all the closed surfaces other than the sphere, it was established by Franks and Handel [FH03] and further extended to the category of Hamiltonian homeomorphisms by Le Calvez [LC06]. In higher dimension, the original conjecture for tori was recently established by Hingston [Hin09] and further generalized to closed symplectically aspherical manifolds by Ginzburg [Gin10]. Further extensions, refinements and related results are contained in [Lon00, Lu09, Maz11, GG10b, GG09, Hei09, Hei11, GG10a].

The general strategy of the proof of the Conley conjecture goes along the following lines. A contractible periodic point with period p corresponds to a contractible p -periodic orbit of an associated Hamiltonian system, which in turn is a critical point of the Hamiltonian action functional in period p . For homological reasons, for each period p one can detect critical points of the action functional with non-trivial local homology in Maslov degree d , where d is the half-dimension of the symplectic manifold. Then, if one assumes that the Hamiltonian diffeomorphism has only finitely many contractible periodic points, one among these points must be strongly homologically visible in degree d for infinitely many periods. This means that for infinitely many periods p , the p -periodic orbit associated to the periodic point has non-trivial local homology in Maslov degree d . Then, the crucial part of the proof comes: the existence of a periodic point that is strongly visible in degree d automatically implies the existence of infinitely many other contractible periodic points.

The first part of the proof, namely the detection of a strongly visible periodic point, can be carried out for the case of tori by means of Morse theory for the Hamiltonian action in a suitable functional setting, whereas for more general symplectic manifolds one has to use Floer theory (see e.g. [Sal99]), a special Morse theory for the Hamiltonian action. The second part of the proof, namely the statement asserting that the presence of a strongly visible periodic point implies the existence of infinitely many other ones, is much harder. For the torus, Hingston [Hin09] argued by a highly non-trivial localization process around the strongly visible periodic point: she showed that one can assume that the associated periodic orbit is stationary and that the Hamiltonian function around the orbit has a very special form. A clever analysis of the local properties of the action at these stationary orbit proved the claim. For more general symplectic manifolds, Ginzburg [Gin10] performed an analogous localization at the strongly visible periodic point, and then used several subtle properties of Floer homology in order to conclude.

The motivation of the current work was to give a simple and more transparent argument for this second part of the proof of the Conley conjecture in the case of Hamiltonian diffeomorphisms of tori that are sufficiently C^1 -close to the identity. Instead of relying on the variational principle associated to the Hamiltonian action functional, we make use of a “discrete” version of it, introduced by Chaperon [Cha84, Cha85] and further investigated by Robbin and Salamon [RS93b]. The functional associated to this variational principle is called discrete symplectic action, and it is a generating family for the graph of the Hamiltonian diffeomorphism.

Our main result is the following.

Theorem 1.1. *In the standard symplectic torus $(\mathbb{T}^{2d}, \omega)$, with $d \geq 1$, every Hamiltonian diffeomorphism ϕ sufficiently C^1 -close to the identity possesses infinitely many contractible periodic points.*

1.1. Organization of the paper. In Section 2 we provide the reader with the necessary background. We introduce the variational principle associated to the discrete symplectic action and discuss the relation between its Morse indices at a critical point and the Maslov index of the associated periodic point of the considered Hamiltonian diffeomorphism. Moreover, we review the definition and the basic properties of local homology groups. Section 3 is the core of this paper: we prove the above mentioned homological vanishing principle and we use it to assert that the existence of a strongly visible periodic point implies the existence of infinitely many other contractible periodic points. Finally, in Section 4, we prove Theorem 1.1.

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2. PRELIMINARIES

Throughout this paper, ϕ will denote a Hamiltonian diffeomorphism on the standard symplectic torus $(\mathbb{T}^{2d}, \omega)$, where $d \geq 1$. We will further assume that ϕ is sufficiently C^1 -close to the identity, so that its graph admits a generating function (see section 2.1). For us, \mathbb{T}^{2d} is the quotient $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ and we will denote the global Darboux coordinates on it as usual by $z = (x, y)$, so that $\omega = dx \wedge dy$. We refer the reader to [MS98, HZ94] for an introduction to symplectic geometry and Hamiltonian dynamics.

2.1. The discrete symplectic action. We are interested in the periodic points of ϕ , namely at those $z_0 \in \mathbb{T}^{2d}$ such that $\phi^p(z_0) = z_0$ for some period $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. By definition of Hamiltonian diffeomorphism, $\phi = \phi_1$ is the time-1 map of a Hamiltonian flow $\phi_t : \mathbb{T}^{2d} \rightarrow \mathbb{T}^{2d}$, and without loss of generality one can always assume ϕ_t to be generated by a Hamiltonian that is 1-periodic in time, in such a way that

$$(2.1) \quad \phi_{t+1} = \phi_t \circ \phi_1.$$

Therefore, there is a one-to-one correspondence between periodic points z_0 of ϕ and periodic orbits $\gamma(t) = \phi_t(z_0)$ of the corresponding Hamiltonian flow. A well-known property of Hamiltonian diffeomorphisms is that, at least on closed symplectically aspherical manifolds (among which are the tori), the fact that a periodic orbit associated to a periodic point of ϕ is contractible does not depend on the choice of the Hamiltonian flow ϕ_t having ϕ as time-1 map, see [Sch00, Prop. 3.1]. Hence, it makes sense to talk about **contractible periodic points** of the Hamiltonian diffeomorphism ϕ .

It is well known that contractible periodic orbits of Hamiltonian systems can be detected by means of a variational principle: they are critical points of the action functional. In the current paper, we will make use of a discrete version of this variational principle, which was introduced by Chaperon [Cha84, Cha85] in the

early 1980s in order to provide a simple proof of the Arnold conjecture for tori. Here, we recall this principle following Robbin and Salamon [RS93b]. Let us lift ϕ as a \mathbb{Z}^{2d} -equivariant Hamiltonian diffeomorphism $\tilde{\phi} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ that is still C^1 -close to the identity, and therefore admits a generating function $\tilde{S} : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ such that

$$(2.2) \quad \begin{aligned} (x_1, y_1) &= \phi(x_0, y_0) \\ &\text{if and only if} \\ \begin{cases} x_1 - x_0 = \partial_y \tilde{S}(x_1, y_0) \\ y_1 - y_0 = -\partial_x \tilde{S}(x_1, y_0) \end{cases} \end{aligned}$$

The function \tilde{S} is invariant by the action of \mathbb{Z}^{2d} , and therefore it descends to a function of the form $S : \mathbb{T}^{2d} \rightarrow \mathbb{R}$ (see [MS98, Lemma 11.1]). From (2.2) we infer that the critical points of S are precisely the contractible fixed points of ϕ .

The iterated Hamiltonian diffeomorphism ϕ^p , where $p > 1$, is not necessarily C^1 -close to the identity, and therefore its graph may not admit a generating function. However, it admits a so-called generating family. A common choice of generating family is the **discrete symplectic action**

$$\tilde{S}^{\times p} : (\mathbb{R}^{2d})^{\times p} = \underbrace{\mathbb{R}^{2d} \times \dots \times \mathbb{R}^{2d}}_{\times p} \rightarrow \mathbb{R},$$

defined by

$$\tilde{S}^{\times p}(\mathbf{z}) := \sum_{j \in \mathbb{Z}_p} \left(\langle y_j, x_j - x_{j+1} \rangle + S(x_{j+1}, y_j) \right).$$

Hereafter, j must be interpreted as a cyclic index, i.e. $j \in \mathbb{Z}_p$, and we write $\mathbf{z} = (z_0, \dots, z_{p-1})$ and $z_j = (x_j, y_j)$. Notice that

$$\tilde{S}^{\times 1} = \tilde{S}.$$

Moreover $\tilde{S}^{\times p}$ is invariant by the diagonal action of \mathbb{Z}^{2d} on $(\mathbb{R}^{2d})^{\times p}$, and therefore it descends to a functional $S^{\times p} : (\mathbb{R}^{2d})^{\times p} / \mathbb{Z}^{2d} \rightarrow \mathbb{R}$. A straightforward computation shows that $[\mathbf{z}] \in (\mathbb{R}^{2d})^{\times p} / \mathbb{Z}^{2d}$ is a critical point of $S^{\times p}$ if and only if $z_{j+1} = \phi(z_j)$ for each $j \in \mathbb{Z}_p$, and therefore if and only if $[z_0] \in \mathbb{T}^{2d}$ is a contractible fixed point of the Hamiltonian diffeomorphism ϕ^p . In particular, critical points of $S^{\times p}$ are in one-to-one correspondence with contractible p -periodic points of the Hamiltonian diffeomorphism ϕ . In the following, in order to simplify the notation we will omit the squared brackets $[\cdot]$.

We wish to apply the machinery of critical point theory, and more specifically Morse theory, to the discrete symplectic action. In order to do this, we need to make sure that the so-called Palais-Smale condition [Pal63] is satisfied: each sequence $\{\mathbf{z}_\alpha \mid \alpha \in \mathbb{N}\}$ such that $S^{\times p}(\mathbf{z}_\alpha) \rightarrow c$ and $|dS^{\times p}(\mathbf{z}_\alpha)| \rightarrow 0$ admits a converging subsequence.

Proposition 2.1. *The discrete symplectic action $S^{\times p}$ satisfies the Palais-Smale condition, and its critical points are contained in a compact subset of its domain.*

Proof. Following McDuff and Salamon [MS98, page 352], let

$$\tau : (\mathbb{R}^{2d})^{\times p} / \mathbb{Z}^{2d} \xrightarrow{\cong} \mathbb{T}^{2d} \times (\mathbb{R}^{2d})^{\times p-1}$$

be the diffeomorphism given by

$$\tau(z_0, \dots, z_{p-1}) = (z_0, z_1 - z_0, z_2 - z_1, \dots, z_{p-1} - z_{p-2}).$$

Then, we have

$$S^{\times p} \circ \tau^{-1}(z_0, \zeta) = \langle P\chi, \mathbf{v} \rangle + B(z_0, \zeta),$$

$$\forall z_0 \in \mathbb{T}^{2d}, \zeta = (\zeta_1, \dots, \zeta_{p-1}) \in (\mathbb{R}^{2d})^{\times p-1},$$

where B is a function of the form

$$B : (\mathbb{T}^{2d})^{\times p} \rightarrow \mathbb{R},$$

P is the $d(p-1) \times d(p-1)$ matrix which can be written as an upper triangular matrix with the upper part filled by blocks of $d \times d$ identity matrices I , i.e.

$$P = \begin{bmatrix} I & I & \cdots & I \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ 0 & \cdots & 0 & I \end{bmatrix},$$

and we write each ζ_j in symplectic coordinates as (χ_j, v_j) , so that $\chi := (\chi_1, \dots, \chi_{p-1})$ and $\mathbf{v} := (v_1, \dots, v_{p-1})$ are vectors in $(\mathbb{R}^d)^{\times p-1}$.

Now, the quadratic form $Q(\zeta) := \langle P\chi, \mathbf{v} \rangle$ is non-degenerate (and moreover it has Morse index $d(n-1)$). Therefore $|dP(\zeta)| \rightarrow \infty$ as $|\zeta| \rightarrow \infty$. On the other hand the function B has bounded C^1 norm, being defined on a compact space. This proves the proposition. \blacksquare

2.2. The Morse indices. Detecting contractible periodic points of ϕ is equivalent to detecting critical points of the functionals $S^{\times p}$ (for all $p \in \mathbb{N}$). However, if z is a critical point of S , then its product

$$z^{\times n} := \underbrace{(z, \dots, z)}_{\times n}$$

is a critical point of $S^{\times n}$ for each $n \in \mathbb{N}$ (this is another way of saying that a fixed point of ϕ is also a fixed point of any iterated composition of ϕ with itself). In order to estimate the number of contractible periodic points of ϕ one must have some criterium to conclude whether a given critical point \mathbf{z} of $S^{\times p}$ is a “genuine” one or it is rather of the form $\mathbf{z} = \mathbf{w}^{\times n}$ for some $n < p$ and some critical point \mathbf{w} of $S^{\times p/n}$. In certain situations, this can be achieved by looking at the Morse indices.

Let \mathbf{z} be a critical point of $S^{\times p}$. We denote by $\text{mor}(\mathbf{z})$ and $\text{nul}(\mathbf{z})$ the Morse index and the nullity of $S^{\times p}$ at the critical point \mathbf{z} . We recall that these are nonnegative integers defined respectively as the dimension of the negative eigenspace and of the kernel of the Hessian of $S^{\times p}$ at \mathbf{z} . The nullity of critical points of the discrete symplectic action is simply given by the geometric multiplicity of 1 as an eigenvalue of the linearized diffeomorphism $d\phi^p$ at the associated periodic point.

Proposition 2.2. *A vector $\mathbf{Z} = (Z_0, \dots, Z_{p-1}) \in (\mathbb{R}^{2d})^{\times p}$ belongs to the kernel of the Hessian of $S^{\times p}$ at a critical point $\mathbf{z} = (z_0, \dots, z_{p-1})$ if and only if*

$$Z_{j+1} = d\phi(z_j)Z_j, \quad \forall j \in \mathbb{Z}_p.$$

In particular

$$(2.3) \quad \text{nul}(\mathbf{z}) = \dim \ker(\text{d}\phi(z_0)^p - I).$$

Proof. If we write each ϕ in symplectic components as (χ, ν) , then by (2.2) the differential of ϕ can be written in matrix form as

$$\text{d}\phi = \begin{bmatrix} \partial_x \chi & \partial_y \chi \\ \partial_x \nu & \partial_y \nu \end{bmatrix},$$

and

$$\begin{aligned} \partial_x \chi &= (I - \partial_{xy} S)^{-1}, \\ \partial_y \chi &= (I - \partial_{xy} S)^{-1} \partial_{yy} S, \\ \partial_x \nu &= -\partial_{xx} S (I - \partial_{xy} S)^{-1}, \\ \partial_y \nu &= I - \partial_{yx} S - (I - \partial_{xy} S)^{-1} \partial_{yy} S. \end{aligned}$$

Given $Z_j = (X_j, Y_j) \in \mathbb{R}^{2d}$, the vector $Z_{j+1} = (X_{j+1}, Y_{j+1}) := \text{d}\phi(z_j)Z_j$ is given by

$$(2.4) \quad \begin{aligned} X_{j+1} &= (I - \partial_{xy} S)^{-1} (X_j + \partial_{yy} S Y_j), \\ Y_{j+1} &= -\partial_{xx} S (I - \partial_{xy} S)^{-1} X_j + (I - \partial_{yx} S - (I - \partial_{xy} S)^{-1} \partial_{yy} S) Y_j, \end{aligned}$$

where all the derivatives of S are meant to be evaluated at the point (x_{j+1}, y_j) . On the other hand, the Hessian of $S^{\times p}$ at \mathbf{z} is given by

$$\begin{aligned} \text{Hess } S^{\times p}(\mathbf{z})[\mathbf{Z}, \mathbf{Z}'] &= \sum_{j \in \mathbb{Z}_p} \langle X_j - (I - \partial_{xy} S) X_{j+1} + \partial_{yy} S Y_j, Y_j' \rangle \\ &\quad + \sum_{j \in \mathbb{Z}_p} \langle \partial_{xx} S X_{j+1} + Y_{j+1} - (I - \partial_{yx} S) Y_j, X_{j+1}' \rangle, \end{aligned}$$

for each $\mathbf{Z} = (Z_0, \dots, Z_{p-1})$ and $\mathbf{Z}' = (Z_0', \dots, Z_{p-1}')$, with $Z_j = (X_j, Y_j)$ and $Z_j' = (X_j', Y_j')$. Then, \mathbf{Z} belongs to the kernel of the Hessian of $S^{\times p}$ at \mathbf{z} if and only if (2.4) holds for each $j \in \mathbb{Z}_p$. Notice that, for such $\mathbf{Z} = (Z_0, \dots, Z_{p-1})$, we have

$$\text{d}\phi(z_0)^p Z_0 = \text{d}\phi(z_{p-1}) \text{d}\phi(z_{p-2}) \dots \text{d}\phi(z_0) Z_0 = Z_0.$$

Hence, the map $\mathbf{Z} \mapsto Z_0$ is a linear isomorphism from the kernel of $\text{Hess } S^{\times p}(\mathbf{z})$ to the kernel of $(\text{d}\phi(z_0)^p - I)$, and (2.3) follows. \blacksquare

In order to use the Morse indices for distinguishing among critical points, it is useful to know how the sequences $\{\text{mor}(\mathbf{z}^{\times n}) \mid n \in \mathbb{N}\}$ and $\{\text{nul}(\mathbf{z}^{\times n}) \mid n \in \mathbb{N}\}$ behave. All we need to know about the nullity is described in the following proposition. We recall that the **Floquet multipliers** of ϕ^p at a fixed point z_0 are defined as the eigenvalues of $\text{d}\phi(z_0)^p$.

Proposition 2.3. *Let \mathbf{z} be a critical point of $S^{\times p}$. For each integer $n > 1$, if none of the Floquet multipliers of ϕ^p at z_0 other than 1 are n -th complex roots of 1, then $\text{nul}(\mathbf{z}) = \text{nul}(\mathbf{z}^{\times n})$.* \blacksquare

Proof. By applying Long’s version of Bott-type formulas for the geometric multiplicity of eigenvectors of symplectic matrices, we infer

$$\dim \ker(d\phi(z_0)^{pn} - I) = \sum_{\lambda \in \sqrt[n]{1}} \dim_{\mathbb{C}} \ker(d\phi(z_0)^p - \lambda I),$$

see [Lon02, Theorem 9.2.1]. Since no Floquet multiplier of ϕ^p at z_0 other than 1 is an n -th complex root of 1, the above equation reduces to

$$\dim \ker(d\phi(z_0)^{pn} - I) = \dim_{\mathbb{C}} \ker(d\phi(z_0)^p - I) = \dim \ker(d\phi(z_0)^p - I).$$

Hence, by Proposition 2.2 our statement follows. \blacksquare

As for the Morse index, one may directly develop an analogous of Bott’s iteration theory [Bot56]. However, we prefer to proceed indirectly by relying on the relation between the Morse index of the discrete symplectic action and the Maslov index of the periodic points of ϕ , and then by using Long’s extensions of Bott’s theory for the Maslov index.

2.3. The Maslov index. The symplectic group $\mathrm{Sp}(2d)$ has an infinite cyclic fundamental group. The Maslov index is an integer $\mathrm{mas}(\Gamma)$ which is associated to elements Γ of the universal cover $\widetilde{\mathrm{Sp}}(2d)$. Here, we regard an element of $\widetilde{\mathrm{Sp}}(2d)$ covering $A \in \mathrm{Sp}(2d)$ as a homotopy class of continuous paths $\Gamma : [0, 1] \rightarrow \mathrm{Sp}(2d)$ joining the identity I with A . The integer $\mathrm{mas}(\Gamma)$ is roughly the number of half-windings made by the path Γ in the symplectic group. In the following we briefly recall its precise definition, and we refer the reader to the books [Abb01, Lon02] for a detailed account.

Let $r : \mathrm{Sp}(2d) \rightarrow U(d)$ be the retraction that sends any $A \in \mathrm{Sp}(2d)$ to the unitary complex matrix obtained from $(AA^T)^{-1/2}A$ after identifying \mathbb{R}^{2d} with \mathbb{C}^d . By composing this retraction with the determinant we obtain the so-called rotation function $\rho := \det \circ r : \mathrm{Sp}(2d) \rightarrow S^1 \subset \mathbb{C}$. Given a path $\Gamma \in \widetilde{\mathrm{Sp}}(2d)$, we define its **average Maslov index** $\overline{\mathrm{mas}}(\Gamma)$ by

$$\overline{\mathrm{mas}}(\Gamma) := \frac{\arg(\rho(\gamma(1))) - \arg(\rho(\gamma(0)))}{\pi}.$$

The reason for this terminology lies the property described later in (2.6). Notice that the average Maslov index is not necessarily an integer.

Now, we denote by $\mathrm{Sp}^*(2d) \subset \mathrm{Sp}(2d)$ the subset of those symplectic matrices that do not have 1 as an eigenvalue. Its complement $\mathrm{Sp}^0(2d) := \mathrm{Sp}(2d) \setminus \mathrm{Sp}^*(2d)$ is a singular hypersurface in the symplectic group that separates $\mathrm{Sp}^*(2d)$ in two connected components. Now, consider the $2d \times 2d$ symplectic diagonal matrices $W' := \mathrm{diag}(2, 1/2, -1, -1, \dots, -1)$ and $W'' := -I = \mathrm{diag}(-1, -1, \dots, -1)$. This two matrices belong to different connected components of $\mathrm{Sp}^*(2d)$, and we have $\rho(W') = (-1)^{d-1} = -\rho(W'')$. Given $\Gamma \in \widetilde{\mathrm{Sp}}(2d)$ with $\Gamma(1) \in \mathrm{Sp}^*(2d)$, let us choose an arbitrary continuous path $\Gamma' : [0, 1] \rightarrow \mathrm{Sp}^*(2d)$ joining $\Gamma(1)$ with either W' or W'' . We denote by $\Gamma * \Gamma' \in \widetilde{\mathrm{Sp}}(2d)$ the path obtained by concatenating Γ and Γ' . Then, the **Maslov index** of Γ is defined as

$$\mathrm{mas}(\Gamma) := \overline{\mathrm{mas}}(\Gamma * \Gamma').$$

The function “mas” is locally constant on the set of those $\Gamma \in \widetilde{\mathrm{Sp}}(2d)$ with $\Gamma(1) \in \mathrm{Sp}^*(2d)$, and it cannot be continuously extended to the whole $\mathrm{Sp}(2d)$. Then, we

extend it as a lower semi-continuous function by setting¹

$$(2.5) \quad \text{mas}(\Gamma) := \liminf_{\substack{\Psi \rightarrow \Gamma \\ \Psi(1) \in \text{Sp}^*(2d)}} \text{mas}(\Psi),$$

where $\Psi \rightarrow \Gamma$ denotes the pointwise convergence.

Now, given $\Gamma \in \widetilde{\text{Sp}}(2d)$ we can define its n -fold iteration $\Gamma^{\times n} \in \widetilde{\text{Sp}}(2d)$ by

$$\Gamma^{\times n} \left(\frac{j+t}{n} \right) = \Gamma(t) \underbrace{\Gamma(1) \dots \Gamma(1)}_{\times n}, \quad \forall j \in \{0, \dots, n-1\}, t \in [0, 1].$$

The Maslov index $\text{mas}(\Gamma^{\times n})$ grows according to the following iteration formula established by Liu and Long [LL98, LL00].

$$(2.6) \quad n \overline{\text{mas}}(\Gamma) - d \leq \text{mas}(\Gamma^{\times n}) \leq n \overline{\text{mas}}(\Gamma) + d - \dim \ker(\Gamma^{\times n}(1) - I).$$

Moreover, if these inequalities are not both strict, then 1 is the only eigenvalue of $\Gamma^{\times n}(1)$. If 1 is the only eigenvalue of $\Gamma(1)$ and $\overline{\text{mas}}(\Gamma) = 0$, then $\text{mas}(\Gamma) = \text{mas}(\Gamma^{\times n})$ for all $n \in \mathbb{N}$. We refer the reader to [Lon02, chapters 9,10] for a proof of these (and many others) iteration properties.

Now, let us consider again the Hamiltonian diffeomorphism $\phi : \mathbb{T}^{2n} \rightarrow \mathbb{T}^{2n}$ and a contractible p -periodic point z_0 of it. Let ϕ_t be a Hamiltonian flow whose time-1 map is ϕ and that is generated by a Hamiltonian that is 1-periodic in time. Consider the path $Z : [0, \infty) \rightarrow \text{Sp}(2d)$ given by $Z(t) = d\phi_t(z_0)$. Let \mathbf{z} be the critical point of $S^{\times p}$ associated to the contractible fixed point z_0 . We define the **average Maslov index** of \mathbf{z} and, for each period $n \in \mathbb{N}$, the **Maslov index** of $\mathbf{z}^{\times n}$ as the integers

$$\begin{aligned} \overline{\text{mas}}(\mathbf{z}) &:= \overline{\text{mas}}(Z|_{[0,p]}), \\ \text{mas}(\mathbf{z}^{\times n}) &:= \text{mas}(Z|_{[0,pn]}), \end{aligned}$$

where the restrictions $Z|_{[0,p]}$ and $Z|_{[0,pn]}$ are seen as elements of $\widetilde{\text{Sp}}(2d)$. It turns out that the Maslov index of \mathbf{z} depends only on z_0 and ϕ , and not on the specific choice of the Hamiltonian flow (this is indeed true in all the closed symplectically aspherical manifolds, see [Sch00, page 440]). By equation (2.1), we have that

$$Z(t+1) = Z(t)Z(1),$$

and therefore we can summarize the above mentioned iteration properties of the Maslov index as follows.

Proposition 2.4. *Let $\mathbf{z} = (z_0, \dots, z_{p-1})$ be the critical point of $S^{\times p}$.*

(i) *For all $n \in \mathbb{N}$ we have*

$$n \overline{\text{mas}}(\mathbf{z}) - d \leq \text{mas}(\mathbf{z}^{\times n}) \leq n \overline{\text{mas}}(\mathbf{z}) + d - \text{nul}(\mathbf{z}^{\times n}).$$

Moreover, if these inequalities are not both strict, then 1 is the only Floquet multiplier of ϕ^{pn} at z_0 .

(ii) *If 1 is the only Floquet multiplier of ϕ^p at z_0 and $\overline{\text{mas}}(\mathbf{z}) = 0$, then $\text{mas}(\mathbf{z}) = \text{mas}(\mathbf{z}^{\times n})$ for all $n \in \mathbb{N}$. ■*

¹For other applications, different extensions may be more suitable. For instance, Robbin and Salamon consider in [RS93a] the average between the maximal lower semi-continuous and the minimal upper semi-continuous extensions.

The Maslov index is related to the Morse index of the discrete symplectic action according to the following statement, which may be regarded as a symplectic analog of the Morse index Theorem.

Proposition 2.5 (Symplectic Morse index Theorem). *For every critical point \mathbf{z} of $S^{\times p}$ we have*

$$(2.7) \quad \text{mor}(\mathbf{z}) = \text{mas}(\mathbf{z}) + \underbrace{\frac{1}{2} \dim(\text{domain}(S^{\times p}))}_{=pd}.$$

Proof. Assume first that the critical point \mathbf{z} of $S^{\times p}$ is non-degenerate. By Proposition 2.2 this is equivalent to the fact that z_0 is a non-degenerate fixed point of the p -fold composition $\phi \circ \dots \circ \phi$. In this case, equation (2.7) was proved by Robbin and Salamon in [RS93b, Th. 4.1].

Now, consider the Morse index as a continuous (and actually locally constant) function on the space of non-degenerate symmetric bilinear forms. Then, its extension to the whole space of symmetric bilinear forms is the maximal lower semi-continuous one. In the same way, in equation (2.5) we defined the Maslov index on the whole $\widetilde{\text{Sp}}(2d)$ as the maximal lower semi-continuous extension of the Maslov index for paths with final endpoint in $\text{Sp}^*(2d)$ (i.e. the “non-degenerate” paths). Hence equation (2.7) still holds when \mathbf{z} is a degenerate critical point of $S^{\times p}$. ■

2.4. Local homology. Let M be a smooth manifold, and p a critical point of a smooth function $f : M \rightarrow \mathbb{R}$ whose critical points are all isolated. If p is non-degenerate, by the Morse Lemma [Mil63, page 6] we know that around p the function f looks like a non-degenerate quadratic form whose negative eigenspace has dimension equal to the Morse index $\text{mor}(p)$. If p is degenerate, i.e. its nullity $\text{nul}(p)$ is non-zero, then the behavior of f around p can be more complicated, as described by the Generalized Morse Lemma [GM69, Lemma 1]. Some properties of the critical point p are captured by its **local homology**, which is defined as the graded group

$$C_*(p) := H_*(\{f < c\} \cup \{p\}, \{f < c\}),$$

where $c = f(p)$ and H_* denotes the singular homology functor. For our purpose, it is enough to consider local homology with coefficients in the field \mathbb{Z}_2 . We refer the reader to [Cha93, Ch. 1] for the main properties of local homology groups. Here, we wish to recall that $C_j(p)$ is always trivial in degrees $j < \text{mor}(p)$ and $j > \text{mor}(p) + \text{nul}(p)$. Moreover, if a strip $\{c' \leq f \leq c''\}$ is “sufficiently compact” (for instance if f satisfies the Palais-Smale condition on it), then we have the generalized Morse inequality

$$(2.8) \quad \text{rank } H_j(\{f < c''\}, \{f < c'\}) \leq \sum_p \text{rank } C_j(p),$$

where the above sum runs over all the critical points p of f with $c' \leq f(p) < c''$.

In the next section, we will need the following property of local homology groups.

Proposition 2.6. *Assume that a critical point p of $f : M \rightarrow \mathbb{R}$ has non-trivial local homology (with coefficients in \mathbb{Z}_2) in maximal degree $k = \text{mor}(p) + \text{nul}(p)$. Then, the local homology of p is concentrated in degree k and it is isomorphic to the*

coefficient field, i.e.

$$(2.9) \quad C_*(p) = C_k(p) \cong \mathbb{Z}_2.$$

Moreover, consider an embedded k -dimensional disk $D \subset M$ containing p in its interior and such that $f|_{D \setminus \{p\}} < f(p)$. Then, the group $C_*(p)$ is generated by $[D]$.

Proof. Since the statement is local we can assume that $M = \mathbb{R}^N$ and p is the origin. Let $m := \text{mor}(p)$ and $n := \text{nul}(p)$, so that $k = m + n$. By the Generalized Morse Lemma we can assume that the function f has the form

$$f(x_0, x_-, x_+) = f_0(x_0) + |x_+|^2 - |x_-|^2, \quad \forall x_0 \in \mathbb{R}^n, x_- \in \mathbb{R}^m, x_+ \in \mathbb{R}^{N-k},$$

where $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function whose Hessian is identically zero at the origin p . Let us denote by $C_*^0(p)$ the local homology of f_0 at p . By Gromoll-Meyer's Shifting Theorem (see [GM69] or [Cha93, page 50]) we have that $C_*(p) \cong C_{*-m}^0(p)$. In particular $C_n^0(p) \cong C_k(p)$ is non-trivial, which in turn implies that p is an isolated local maximum for the function f_0 . However, the local homology of local maxima is concentrated in maximal degree n , where it is isomorphic to the coefficient field \mathbb{Z}_2 (see [Cha93, page 51]). This proves (2.9).

Now, consider a disk D as in the statement of the lemma, and set

$$\mathbb{E}^+ := \{(x_0, x_-, x_+) \in \mathbb{R}^N \mid x_0 = 0, x^- = 0\}.$$

We claim that D is transverse to \mathbb{E}^+ at the origin p . Indeed, assume that there is a non-zero $v \in T_p D \cap \mathbb{E}^+$. Consider a smooth curve $\gamma : (-\epsilon, \epsilon) \rightarrow D$ such that $\dot{\gamma}(0) = v$. Then, we have

$$\frac{d^2}{dt^2} \Big|_{t=0} f \circ \gamma(t) = \frac{d}{dt} \Big|_{t=0} df(\gamma(t))\dot{\gamma}(t) = \underbrace{\text{Hess } f(p)[v, v]}_{>0} + \underbrace{df(p)\dot{\gamma}(0)}_{=0} > 0,$$

which contradicts the fact that $f < 0$ on $D \setminus \{p\}$.

Set $B_r := \{(x_0, x_-) \in \mathbb{R}^k \mid |x_0|^2 + |x_-|^2 \leq r\}$. By the Implicit Function Theorem, there exist $r > 0$ and a smooth map $\psi : B_r \rightarrow \mathbb{R}^{N-k}$ such that the disk

$$D' := \{(x_0, x_-, \psi(x_0, x_-)) \mid (x_0, x_-) \in B_r\}.$$

is a compact neighborhood of the origin p in D . Notice that $[D'] = [D]$ in $C_k(p)$. Up to choosing a smaller $r > 0$, we can further assume that $f_0(x_0) < f_0(p)$ for all the $x_0 \in \mathbb{R}^n \setminus \{p\}$ with $|x_0| \leq r$. Let $h_t : D' \hookrightarrow \mathbb{R}^N$ (for $0 \leq t \leq 1$) be the isotopy given by

$$h_t(x_0, x_-, x_+) = (x_0, x_-, (1-t)x_+),$$

and set $D'' := h_1(D')$. Notice that D'' is a generator of the local homology $C_k(p)$. Moreover, f is a Lyapunov function for h_t , i.e.

$$\frac{d}{dt} f \circ h_t \leq 0.$$

Hence $[D''] = [D']$ in $C_k(p)$. This completes the proof. \blacksquare

3. STRONGLY VISIBLE FIXED POINTS

An isolated contractible fixed point z of ϕ is called **strongly visible** when there exists an infinite subset $\mathbb{K} \subseteq \mathbb{N}$ such that

$$(3.1) \quad C_{dn+d}(z^{\times n}) \neq 0, \quad \forall n \in \{1\} \cup \mathbb{K}.$$

If we consider the ‘‘Maslov grading’’ for the local homology groups, i.e. if we shift the usual ‘‘Morse grading’’ by $-\frac{1}{2} \dim(\text{domain}(S^{\times n})) = -dn$ according to Proposition 2.5, then the critical point $z^{\times n}$ of $S^{\times n}$ is ‘‘homologically visible’’ in the same degree d for all $n \in \{1\} \cup \mathbb{K}$. The presence of a strongly visible fixed point z forces the existence of infinitely many other contractible periodic points whose average action accumulates at the action of z , according to the following theorem.

Theorem 3.1. *If a Hamiltonian diffeomorphism $\phi : \mathbb{T}^{2d} \rightarrow \mathbb{T}^{2d}$ has a strongly visible fixed point z , then it admits an infinite sequence of distinct contractible periodic points $\{z_\alpha \mid \alpha \in \mathbb{N}\}$ with the following property: if z_α is the critical point of $S^{\times p_\alpha}$ corresponding to z_α , then*

$$\frac{1}{p_\alpha} S^{\times p_\alpha}(z_\alpha) \searrow S(z) \quad \text{as } \alpha \rightarrow \infty.$$

Theorem 3.1 is a consequence of the following homological vanishing property, which is inspired by an analogous statement due to Bangert and Klingenberg [BK83, Th. 2] in the setting of closed geodesics.

Lemma 3.2 (Homological vanishing). *Let z be a strongly visible fixed point of the Hamiltonian diffeomorphism $\phi : \mathbb{T}^{2d} \rightarrow \mathbb{T}^{2d}$. Set $c := S(z)$ and fix $\epsilon > 0$. Then, the inclusion-induced homomorphism*

$$\iota_* : C_*(z^{\times n}) \rightarrow H_*(\{S^{\times n} < nc + \epsilon\}, \{S^{\times n} < nc\})$$

is trivial, provided $n \in \mathbb{K}$ is big enough.

Proof. Assumption (3.1) implies that $\text{mor}(z^{\times n}) \leq dn + d \leq \text{mor}(z^{\times n}) + \text{nul}(z^{\times n})$ for all $n \in \{1\} \cup \mathbb{K}$. By the symplectic Morse index Theorem (Proposition 2.5) we infer

$$\text{mas}(z^{\times n}) \leq d \leq \text{mas}(z^{\times n}) + \text{nul}(z^{\times n}), \quad \forall n \in \{1\} \cup \mathbb{K}.$$

This inequality, together with Proposition 2.4(i), implies that $\overline{\text{mas}}(z) = 0$ and therefore $\text{mas}(z^{\times n}) + \text{nul}(z^{\times n}) \leq d$ for all $n \in \mathbb{N}$. Hence

$$\text{mor}(z^{\times n}) + \text{nul}(z^{\times n}) = dn + d, \quad \forall n \in \{1\} \cup \mathbb{K}.$$

By Proposition 2.6 we infer that the local homology $C_*(z^{\times n})$ is concentrated in maximal degree $dn + d$ and it is equal to the coefficient group \mathbb{Z}_2 . Notice that $C_{2d}(z) \neq 0$, and therefore z is an isolated local maximum of S .

In the remaining of the proof, we will work inside a fundamental domain of the universal cover of $(\mathbb{R}^{2d})^{\times n} / \mathbb{Z}^{2d}$. Hence we can assume that the domain of $S^{\times n}$ is simply $(\mathbb{R}^{2d})^{\times n}$ without quotienting by \mathbb{Z}^{2d} . Moreover, let us assume without loss of generality that $z = 0$ and $c = S(0) = 0$. Let $R > 0$ be sufficiently small so that $S < 0$ outside the origin in a ball of radius $3R$ centered at 0. For $n \in \mathbb{K}$, consider

the $(dn + d)$ -dimensional vector subspace

$$\mathbb{E}_n = \left\{ \mathbf{z} = (x_0, y_0, \dots, x_{n-1}, y_{n-1}) \mid \begin{array}{l} x_0, \dots, x_{n-1}, w \in \mathbb{R}^d \\ y_j = w + x_{j+1} - x_j \quad \forall j \in \mathbb{Z}_n \end{array} \right\},$$

and, for $0 < r < R$, the polydisc

$$W_n = W_n(R, r) = \left\{ \mathbf{z} \in \mathbb{E}_n \mid |x_0| \leq R, \dots, |x_{n-1}| \leq R, |w| \leq r \right\}.$$

On W_n , the discrete action functional $S^{\times n}$ is given by

$$S^{\times n}(\mathbf{z}) = \sum_{j \in \mathbb{Z}_n} \left(-|x_j - x_{j+1}|^2 + S(x_{j+1}, w + x_{j+1} - x_j) \right), \quad \forall \mathbf{z} \in W_n.$$

By Proposition 2.6, $[W_n]$ is a generator of the local homology of $S^{\times n}$ at $0^{\times n}$.

Now, fix a vector $v \in \mathbb{R}^d$ such that $|v| = r$, and let $h : [0, 1] \times W_n \rightarrow (\mathbb{R}^{2d})^{\times n}$ be the homotopy given by

$$h(t, \mathbf{z}) = \mathbf{z} + t\mathbf{z}',$$

where $\mathbf{z} = (x_0, y_0, \dots, x_{n-1}, y_{n-1})$, $\mathbf{z}' = (0, y'_0, 0, y'_1, \dots, 0, y'_{n-1})$, and

$$\begin{aligned} y'_j &= 0, & \forall j \in \{0, \dots, n' - 1\}, \\ y'_j &= v, & \forall j \in \{n', \dots, n - 1\}. \end{aligned}$$

Notice that

$$\begin{aligned} (3.2) \quad S^{\times n} \circ h(t, \mathbf{z}) &= \sum_{j=0}^{n'-1} \left(-|x_j - x_{j+1}|^2 + S(x_{j+1}, w + x_{j+1} - x_j) \right) \\ &+ \sum_{j=n'}^{n-1} \left(-|x_j - x_{j+1}|^2 + S(x_{j+1}, tv + w + x_{j+1} - x_j) \right) \\ &+ t \langle v, x_{n'} - x_0 \rangle. \end{aligned}$$

Let $\rho : [0, R] \rightarrow [0, \infty)$ be a monotone increasing continuous function such that

$$\rho(s) \leq \min \left\{ s^2, \min_{|\mathbf{z}''|=s} \{-S(\mathbf{z}'')\} \right\}.$$

This function allows us to express the following estimate:

$$-|x_j - x_{j+1}|^2 + S(x_{j+1}, u + x_{j+1} - x_j) \leq -\rho\left(\frac{|u|}{2}\right), \quad \forall u \in \mathbb{R}^d \text{ with } |u| \leq R.$$

Now, let us apply this estimate to (3.2). Notice that, since $|v| = r \geq |w|$, at least one of the vectors w and $v + w$ has norm larger than or equal to $r/2$. Hence, if we choose $r < \epsilon/(2R)$, $n' > 2Rr/\rho(\frac{r}{4})$ and $n > 2n'$, we have

$$\begin{aligned} S^{\times n} \circ h(t, \mathbf{z}) &\leq t \langle v, x_{n'} - x_0 \rangle \leq t |v| |x_{n'} - x_0| \leq 2Rr < \epsilon, \\ S^{\times n} \circ h(1, \mathbf{z}) &\leq -n' \rho\left(\frac{r}{4}\right) + \underbrace{\langle v, x_{n'} - x_0 \rangle}_{\leq |v|(|x_{n'}| + |x_0|)} \leq -n' \rho\left(\frac{r}{4}\right) + 2Rr < 0. \end{aligned}$$

This proves that the homotopy h deforms the disk W_n into the action sub-level $\{S^{\times n} < 0\}$, and the disk W_n remains in the action sub-level $\{S^{\times n} < \epsilon\}$ along the homotopy. In order to conclude the proof it is enough to show that, up to further assuming $r < \rho(R)/2R$, the boundary ∂W_n of the disk remains in the action sub-level $\{S^{\times n} < 0\}$ along the homotopy. Consider an arbitrary $\mathbf{z} \in \partial W_n$. We have

two (non-exclusive) cases: $|w| = r$ or $|x_j| = R$ for some $j \in \mathbb{Z}_n$. In the former case, $|w| = r$, we have

$$\begin{aligned} S^{\times n} \circ h(t, \mathbf{z}) &\leq -n' \rho\left(\frac{|w|}{2}\right) + t \langle v, x_{n'} - x_0 \rangle \\ &\leq -n' \rho\left(\frac{r}{2}\right) + 2Rr \\ &\leq -n' \rho\left(\frac{r}{4}\right) + 2Rr \\ &< 0. \end{aligned}$$

In the latter case, $|x_j| = R$ for some $j \in \mathbb{Z}_n$, we have

$$S^{\times n} \circ h(t, \mathbf{z}) \leq \underbrace{S(x_j, y_{j-1} + ty'_{j-1})}_{\leq -\rho(R)} + t \underbrace{\langle v, x_{n'} - x_0 \rangle}_{\leq 2Rr} \leq -\rho(R) + 2Rr < 0. \quad \blacksquare$$

Proof of Theorem 3.1. By contradiction, let us assume that such a sequence of periodic points does not exist. Hence there exists $\epsilon > 0$ such that, for each $n \in \mathbb{K}$, there are no critical values of $S^{\times n}$ in the interval $(nc, nc + \epsilon]$. This implies that the inclusion induces a monomorphism

$$\iota_* : C_*(z^{\times n}) \hookrightarrow H_*(\{S^{\times n} < nc + \epsilon\}, \{S^{\times n} < nc\}),$$

which contradicts the homological vanishing principle of Lemma 3.2. \blacksquare

4. THE CONLEY CONJECTURE

In this section we carry out the proof of Theorem 1.1 stated in the introduction. By Proposition 2.1, for each $n \in \mathbb{N}$ there exists $c > 0$ such that all the critical values of the discrete symplectic action $S^{\times n}$ are contained in the interval $(-nc, nc)$. Following McDuff and Salamon as in the proof of Proposition 2.1, after a suitable reparametrization we can assume that the domain of $S^{\times n}$ is $\mathbb{T}^{2d} \times (\mathbb{R}^{2d})^{\times n-1}$, and

$$S^{\times n}(z_0, \zeta) = Q(\zeta) + B(z_0, \zeta), \quad \forall z_0 \in \mathbb{T}^{2d}, \zeta \in (\mathbb{R}^{2d})^{\times kn-1}.$$

Here $Q(\zeta) = \langle Q'\zeta, \zeta \rangle$ is a non-degenerate quadratic form with Morse index $d(n-1)$, while B is a function of the form $B : (\mathbb{T}^{2d})^{\times kn} \rightarrow \mathbb{R}$. Let us consider the orthogonal spectral decomposition associated to the symmetric matrix Q' , i.e.

$$(\mathbb{R}^{2d})^{\times kn-1} = \mathbb{E}^+ \oplus \mathbb{E}^-,$$

where \mathbb{E}^+ [resp. \mathbb{E}^-] is the positive [resp. negative] eigenspace of Q' . Since the Morse index of Q is $d(n-1)$ we have

$$\dim(\mathbb{E}^+) = \dim(\mathbb{E}^-) = d(n-1).$$

For each $r > 0$ we define

$$N(r) := \{(z_0, \zeta) \in \mathbb{T}^{2d} \times (\mathbb{R}^{2d})^{\times n-1} \mid Q(\zeta) \leq r\},$$

$$L(r) := \{(z_0, \zeta) \in N(r) \mid Q(\zeta) \leq -r\}.$$

By the Künneth formula and excision, for each $r > 0$ we have

$$\begin{aligned} H_j(N(r), L(r)) &\cong \bigoplus_{k \in \mathbb{Z}} H_{j-k}(\mathbb{T}^{2d}) \otimes H_k(\{Q \leq r\}, \{Q \leq -r\}) \\ &\cong \bigoplus_{k \in \mathbb{Z}} H_{j-k}(\mathbb{T}^{2d}) \otimes H_k(D^{d(n-1)}, \partial D^{d(n-1)}) \\ &\cong H_{j-d(n-1)}(\mathbb{T}^{2d}), \end{aligned}$$

where $D^{d(n-1)}$ is the unit disk in \mathbb{E}^- . In particular

$$(4.1) \quad \mathbf{H}_{dn+d}(N(r), L(r)) \cong \mathbf{H}_{2d}(\mathbb{T}^{2d}) \neq 0.$$

For $r > 0$ large enough, the set $N(r)$ contains all the critical points of $S^{\times n}$, $L(r)$ contains no critical points of $S^{\times n}$, and the gradient of $S^{\times n}$ is transverse to the boundaries of $N(r)$ and $L(r)$. Hence, we can use the gradient flow of $S^{\times n}$ to deform $N(r)$ onto the sub-level $\{S^{\times n} \leq nc\}$ and $L(r)$ onto the sub-level $\{S^{\times n} \leq -nc\}$ respectively. This, together with (4.1), implies

$$\mathbf{H}_{dn+d}(\{S^{\times n} < nc\}, \{S^{\times n} < -nc\}) \cong \mathbf{H}_{dn+d}(N(r), L(r)) \neq 0.$$

By the Morse inequality (see (2.8)) we infer that there exists a critical point $z(n)$ of $S^{\times n}$ such that

$$(4.2) \quad \mathbf{C}_{dn+d}(z(n)) \neq 0.$$

Now, if the sequence $\{z(n) \mid n \in \mathbb{N}\}$ corresponds to infinitely many periodic points of ϕ , we are done. Hence, let us assume that this sequence corresponds to only finitely many periodic points. Moreover, let us assume that ϕ has only finitely many contractible fixed points.

Choose a prime number p' larger than the maximum among the basic periods of those contractible periodic points of ϕ . Let $F \subset \mathbb{C}$ be the (finite) subset of the Floquet multipliers of the contractible fixed points of ϕ , and consider $F' = \{q \in \mathbb{Q} \mid e^{i2\pi q} \in F\}$. Then choose a prime number p larger than or equal to p' and larger than the maximum among the denominators of the rational numbers in F' . By Proposition 2.3, for each contractible fixed point z of ϕ , we have

$$(4.3) \quad \text{nul}(z) = \text{nul}(z^{\times n}), \quad \forall n \in \mathbb{K}_p := \{p^k \mid k \in \mathbb{N}\}.$$

By our assumptions on $\{z(n) \mid n \in \mathbb{N}\}$ and p , there exists a contractible fixed point z of ϕ and an infinite subset $\mathbb{K}'_p \subseteq \mathbb{K}_p$ such that $z(n) = z^{\times n}$ for all $n \in \mathbb{K}'_p$. All we have to do in order to conclude the proof is to show that $\mathbf{C}_{2d}(z) \neq 0$, namely that z is an isolated local maximum of S . Indeed, this would prove that z is a strongly visible fixed point, and Theorem 1.1 would follow from Theorem 3.1.

By (4.2) we infer

$$\text{mor}(z^{\times n}) \leq dn + d \leq \text{mor}(z^{\times n}) + \text{nul}(z^{\times n}), \quad \forall n \in \mathbb{K}'_p,$$

and therefore

$$(4.4) \quad \text{mas}(z^{\times n}) \leq d \leq \text{mas}(z^{\times n}) + \text{nul}(z^{\times n}), \quad \forall n \in \mathbb{K}'_p.$$

Proposition 2.4(i) implies that $\overline{\text{mas}}(z) = 0$ and $\text{mas}(z^{\times n}) + \text{nul}(z^{\times n}) \leq d$. Hence $\text{mas}(z^{\times n}) + \text{nul}(z^{\times n}) = d$ and

$$(4.5) \quad \text{mor}(z^{\times n}) + \text{nul}(z^{\times n}) = dn + d, \quad \forall n \in \mathbb{K}'_p.$$

Proposition 2.4(i) further implies that 1 is the only Floquet multiplier of ϕ^n at z , and by our choice of the prime p we also infer that 1 is the only Floquet multiplier of ϕ at z . By Proposition 2.4(ii), we obtain that

$$(4.6) \quad \text{mas}(z) = \text{mas}(z^{\times n}), \quad \forall n \in \mathbb{N}.$$

Let $C_1 \subset \mathbb{T}^{2d}$ be a central manifold of the vector field ∇S at z . We recall that C_1 is an invariant submanifold for the gradient flow of S such that $\mathbf{T}_z C_1$ is the kernel of the Hessian of S at z , see [GH90, section 3.2]. In particular $\dim(C_1) = \text{nul}(z)$.

Fix $n \in \mathbb{K}'_p$, and consider the diagonal embedding $\psi^{\times n} : \mathbb{T}^{2d} \hookrightarrow (\mathbb{R}^{2d})^{\times n} / \mathbb{Z}^{2d}$ given by $\psi^{\times n}(w) = w^{\times n}$. A straightforward calculation shows that

$$\nabla S^{\times n} \circ \psi^{\times n} = \psi^{\times n} \circ \nabla S,$$

where the gradients in this equation are with respect to the standard flat metrics of $(\mathbb{R}^{2d})^{\times n} / \mathbb{Z}^{2d}$ and \mathbb{T}^{2d} . This implies that the manifold $C_n := \psi^{\times n}(C_1)$ is invariant under the gradient flow of $S^{\times n}$. Moreover, by (4.3), we have

$$\dim(C_n) = \dim(C_1) = \text{nul}(z) = \text{nul}(z^{\times n}),$$

and therefore $\mathbb{T}_{z^{\times n}} C_n$ is the kernel of the Hessian of $S^{\times n}$ at $z^{\times n}$. This proves that C_n is a central manifold of the vector field $\nabla S^{\times n}$ at $z^{\times n}$. Set $c := S(z)$, so that $nc = S^{\times n}(z^{\times n})$. By Gromoll-Meyer's splitting theorem (see [GM69, section 3]), the local homology satisfies

$$\begin{aligned} C_{j+\text{mor}(z^{\times n})}(z^{\times n}) &\cong H_j(\{S^{\times n}|_{C_n} < nc\} \cup \{z^{\times n}\}, \{S^{\times n}|_{C_n} < nc\}) \\ (4.7) \quad &\cong H_j(\{S|_{C_1} < c\} \cup \{z\}, \{S|_{C_1} < c\}) \\ &\cong C_{j+\text{mor}(z)}(z). \end{aligned}$$

By (4.6) and Proposition 2.5 we have that

$$\text{mor}(z^{\times n}) = \text{mas}(z^{\times n}) + dn = \text{mas}(z) + dn = \text{mor}(z) + d(n-1).$$

Finally this, together with (4.7) and (4.2), implies $C_{2d}(z) \cong C_{dn+d}(z^{\times n}) \neq 0$. ■

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